Discrete Actuator Array Vectorfield Design for Distributed Manipulation

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Abstract

The Modular Distributed Manipulator System (MDMS) is a macroscopic actuator array which can manipulate objects in the plane. The piecewise-constant dynamics of manipulation on the MDMS are developed based on an exact discrete representation of the system. The resulting dynamics are inverted enabling the calculation of an open-loop vector field which provides arbitrary uniform object dynamics. The vector field positions, and under certain assumptions, orients objects.

1 Introduction

The *Modular Distributed Manipulator System* (MDMS) is a macroscopic actuator array which transfers, as well as manipulates, objects in the plane, enhancing applications such as flexible manufacturing and package handling systems. This system has been described in detail in previous work [4, 6][1]. Essentially, the MDMS comprises an fixed array of actuators (cells) each of which is an orthogonally mounted pair or roller wheels whose combined motion provides a directable traction force to an object resting on top. In this system, several cells support a single object that can be made to translate and rotate in the plane. Since sensing and actuation are distributed, each of many objects can be manipulated independently. (Figure 1). An 18 cell prototype is currently in operation.

In this paper, we derive the dynamics of motion of an object on a two-dimensional array of cells. The macroscopic scale of the system requires us to explicitly model the distribution of weight among the discrete set of supports as well as the traction forces.

[1]The MDMS is formerly the Virtual Vehicle

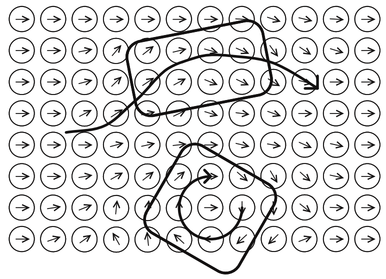

Figure 1: The MDMS: Several objects can be translated and rotated, independently.

Section 2 describes some of the prior work by others and relates it to this work and the prior work of the authors. The dynamic equations derived in Section 3 determine the motion of an object given a set of wheel speeds of each cell. In Section 4, we then solve the inverse problem: determine the necessary wheel speeds to effect desired motion. We design an open-loop wheel velocity field which brings an object to a particular position by inverting the dynamic equations. We show that under certain assumptions this field also brings an object to a particular orientation. Concluding remarks are made in Section 5.

2 Prior Work

Böhringer, Donald, et. al. [1] applied sensorless manipulation ideas of Goldberg [2] to an array of micromechanical actuators which were used to position an orient very small objects to one of a finite number of orientations. Kavraki [3] supplied further analysis of microactuated systems using elliptical potential fields to orient to a single orientation. Due to the small scale of their applications, both Böhringer and Kavraki made continuous field assumptions in their analysis. On the MDMS, however, a smaller number of cells support an object, requiring explicit discrete modeling of the system.

In a previous paper by the authors [4] the first step was taken in examining the dynamics of an object carried by the MDMS, where the one dimensional motion of the object along the array of cells was considered. In that paper, the forces between each cell and the object were calculated, and both a coulomb and a viscous-like friction law were considered. The resulting object dynamics are that of a simple or damped harmonic oscillator, where the frequency, center of oscillation, and damping constant are parameters which change as the object shifts from one set of supports to another. This simple oscillator behavior was also observed in the prototype system. This work was extended into two dimensions in a more recent paper by the authors [5]. Translational forces and rotational torques are calculated as a function of object position. Similar mass-spring-damper behavior in the plane was observed.

This paper refines the authors' previous work in two dimensions by significantly reducing the amount of calculation necessary to calculate forces and torques. A new result presented here is that this refinement allows for the inversion of the dynamics. A wheel velocity field is thus generated which produces desired object dynamics with a single equilibrium position regardless of symmetry and cell resolution, and, under certain assumptions, a single orientation (within symmetry) to the resolution of the cells.

3 Dynamics of Manipulation

Initially, we will consider the dynamics of an array of cells transporting and rotating an object in the plane while it rests on a single arbitrary set of cells. For this, the following assumptions are made:

- Each orthogonally oriented pair of wheels acts as a single support.
- Supports act as springs to support the object.
- The bottom of the object is flat.
- The speed of each wheel is constant.
- Horizontal force between each wheel and the object is due to sliding friction.
- Viscous friction (proportional to speed) exists between the wheels and object which is also proportional to the normal force.

The computation of the horizontal translation and rotation dynamics of the object first requires the use of the equilibrium of the object and constitutive relations for the supports. The horizontal forces and torques are computed using a friction law between the object and wheels. This results in a net force and torque acting on the object as a function of the object's position.

Notation: Normal math font represents scalar variables (e.g. s), arrowed normal math font represents vectors (e.g. $\vec{v}$), and bold font represents matrices (e.g. $\mathbf{m}$). Subscripts x and y indicate x and y components, and subscript i indicates the ith cell. For example, $\mathbf{V}$ is a matrix made up of velocity (column) vectors $\vec{V}_i$ for each cell, with components V_{x_i} and V_{y_i}. $\mathbf{V}$ can also be said to be made up of component (row) vectors $\vec{V}_x$ and $\vec{V}_y$ listing the velocity components for all the cells.

3.1 Normal Forces

Solving for the n forces supporting the object requires the consideration of the equilibrium of the object in both the vertical (z) direction and in rotation about the x and y axes. Consider n cells arranged arbitrarily in the x-y plane having coordinates as entries of the matrix below.

$$\mathbf{X} = \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix} = \begin{bmatrix} x_1 \dots x_n \\ y_1 \dots y_n \end{bmatrix} = \left[\vec{X}_1 \dots \vec{X}_n \right] \tag{1}$$

An object of weight W, whose center of mass is located at $\vec{X}_{cm} = \left[\begin{array}{cc} x_{cm} & y_{cm} \end{array} \right]^{\mathrm{T}}$ resting on n of these cells, is supported by vertical normal forces $\vec{N} = \left[\begin{array}{ccc} N_1 & \dots & N_n \end{array} \right]$. Vertical equilibrium of the object requires that

$$\sum_{i=1}^{n} N_i = W = \left[\begin{array}{ccc} 1 & \dots & 1 \end{array} \right] \vec{N}^{\mathrm{T}}. \tag{2}$$

Rotational equilibrium about the x and y axes requires that the moments induced by the normal forces about any point (in this case, the arbitrarily located origin of our coordinate system) sum to the moment about this point induced by the weight of the object. Therefore,

$$\sum_{i=1}^{n} N_i y_i = \vec{y}\, \vec{N}^{\mathrm{T}} = W y_{cm}, \text{ and} \tag{3}$$

$$\sum_{i=1}^{n} N_i x_i = \vec{x}\, \vec{N}^{\mathrm{T}} = W x_{cm}. \tag{4}$$

At this point in the development, there are n unknowns (the elements of $\vec{N}$), but only three equations

(2,3, and 4) from equilibrium. To solve for the remaining $n-3$ forces, flexibility in the system must be considered. Each support is assumed to be a spring, with Hooke's Law ($N_i = K_s \Delta z_i$) representing the compression of the ith cell under a normal load. Physically, this flexibility is either a flexible suspension under each wheel or, as in the prototype, flexibility in the surface of the bottom of the object.

Assuming the bottom of the object is nominally flat, the flexible cells conform to the bottom of the object to distribute the weight. All the supporting cells lie in this plane, which constrains the normal forces:

$$N_i + a x_i + b y_i + c = 0 \tag{5}$$

Equations 2, 3, and 4 along with n instances of Equation 5 supply $n+3$ equations and $n+3$ unknowns (n N_i's and 3 plane parameters, a, b, and c). The matrix form of this system of equations is

$$\underbrace{\left[\begin{array}{ccc|ccc} 1 & \dots & 0 & 1 & x_1 & y_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 1 & x_n & y_n \\ \hline 1 & \dots & 1 & 0 & 0 & 0 \\ x_1 & \dots & x_n & 0 & 0 & 0 \\ y_1 & \dots & y_n & 0 & 0 & 0 \end{array}\right]}_{\mathbf{A}} \underbrace{\left[\begin{array}{c} N_1 \\ \vdots \\ N_n \\ \hline c \\ b \\ a \end{array}\right]}_{\vec{N}^{\mathrm{T}}_{abc}} = \underbrace{\left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline W \\ Wx_c \\ Wy_c \end{array}\right]}_{\vec{W}} . \tag{6}$$

$\mathbf{A}$ can be inverted to solve for $\vec{N}_{abc}$ (which contains $\vec{N}$ and a, b, and c.) Define a matrix $\mathbf{B}$.

$$\mathbf{B} = \begin{bmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{bmatrix} \tag{7}$$

such that the expression for the matrix $\mathbf{A}$ is

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{I}_{n\times n} & \mathbf{B}^{\mathrm{T}} \\ \hline \mathbf{B} & \mathbf{0}_{3\times 3} \end{array}\right] . \tag{8}$$

The inverse of the matrix $\mathbf{A}$ exists if $\mathbf{B}$ has rank 3 (which is true as long as all the cells do not lie on a line). The expression for the inverse is

$$\mathbf{A}^{-1} = \left[\begin{array}{c|c} \mathbf{I}_{n\times n} - \mathbf{B}^{\mathrm{T}}\left(\mathbf{B}\mathbf{B}^{\mathrm{T}}\right)^{-1}\mathbf{B} & \mathbf{B}^{\mathrm{T}}\left(\mathbf{B}\mathbf{B}^{\mathrm{T}}\right)^{-1} \\ \hline \left(\mathbf{B}\mathbf{B}^{\mathrm{T}}\right)^{-1}\mathbf{B} & -\left(\mathbf{B}\mathbf{B}^{\mathrm{T}}\right)^{-1} \end{array}\right] . \tag{9}$$

Multiplying $\mathbf{A}$ by $\mathbf{A}^{-1}$ verifies the result.

Since $\vec{W}$ only multiplies nonzero elements into the right side of $\mathbf{A}^{-1}$, and only the slopes (a, b, and c) result from the lower portion of $\mathbf{A}^{-1}$, the calculation of $\vec{N}$ uses only the upper right partition of $\mathbf{A}^{-1}$.

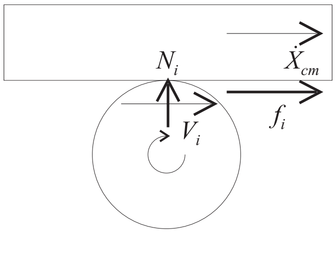

Figure 2: Interaction between wheel and object.

$$\begin{aligned} \vec{N}^{\mathrm{T}} &= \mathbf{B}^{\mathrm{T}}\left(\mathbf{B}\mathbf{B}^{\mathrm{T}}\right)^{-1} \begin{bmatrix} W \\ Wy_{cm} \\ Wy_{cm} \end{bmatrix} \\ &= W\mathbf{B}^{\mathrm{T}}\left(\mathbf{B}\mathbf{B}^{\mathrm{T}}\right)^{-1} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{X}_{cm} \right) \end{aligned} \tag{10}$$

3.2 Planar Dynamics

The full planar dynamics involve translation and rotation of the object. The horizontal forces are derived from the normal forces through the use of a viscous-type friction law (see Figure 2). The horizontal force from each cell $\vec{f}_i$ is proportional to a coefficient of friction μ, that cell's normal force N_i, and the vector difference between the velocity of the wheel and the velocity of the object at the point of the cell. This velocity difference is a function of both the translational velocity of the object $\dot{\vec{X}}_{cm}$, the velocity of the wheel $\vec{V}_i$, the rotation speed of the object about its center of mass ω, and the position difference between the cell and the center of mass $\vec{X}_i - \vec{X}_{cm}$.

$$\vec{f}_i = \mu \left(\vec{V}_i - \dot{\vec{X}}_{cm} + \omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \left(\vec{X}_i - \vec{X}_{cm} \right) \right) N_i \tag{11}$$

The horizontal force from each cell is summed up over all the cells. Define a wheel velocity matrix $\mathbf{V}$ as

$$\mathbf{V} = \begin{bmatrix} V_{1_x} & V_{2_x} & \dots & V_{n_x} \\ V_{1_y} & V_{2_y} & \dots & V_{n_y} \end{bmatrix} . \tag{12}$$

Summing vectorially, the net horizontal force is

$$\vec{f} = \mu \mathbf{V}\ \vec{N}^{\mathrm{T}} - \mu \dot{\vec{X}}_{cm} W. \tag{13}$$

Observe that the net horizontal force is not a function of the object's rotation speed - the terms multiplying ω are identically zero[6]. Furthermore, the second term in this equation is a dissipative linear damping term. The substitution of $\vec{N}$ from Equation 10 into Equation 13 yields

$$\vec{f} = \underbrace{\mu W \mathbf{V}\mathbf{B}^{\mathrm{T}} \left(\mathbf{B}\mathbf{B}^{\mathrm{T}}\right)^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{k_s}} \vec{X}_{cm} + \underbrace{\mu W \mathbf{V}\mathbf{B}^{\mathrm{T}} \left(\mathbf{B}\mathbf{B}^{\mathrm{T}}\right)^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{f}_o} - \mu \dot{\vec{X}}_{cm} W \quad (14)$$

where $\mathbf{k_s}$ is a constant 2×2 matrix and $\vec{f}_o$ is a constant 2×1 vector. The matrix $\mathbf{k_s}$ is essentially a matrix of spring constants, since it specifies force as a linear function of position. The vector $\vec{f}_o$ is an offset force.

In two dimensions, the torque each cell applies to the object is the scalar cross product of the position vector of the point of application of the force, $\vec{X}_i$, relative to the object center of mass, $\vec{X}_{cm}$, and the horizontal force vector from that point, $\vec{f}_i$. After some algebra[6] the total torque on the object is

$$\tau = \mu \vec{R}\vec{N} - \mu \vec{X}_{cm} \times \mathbf{V}\vec{N} - \omega\mu \left(\vec{\mathcal{X}}\vec{N} - W \vec{X}_{cm}^{\mathrm{T}} \vec{X}_{cm}\right) \quad (15)$$

where $R_i = \vec{X}_i \times \vec{V}_i$ defines $\vec{R}$ and $\mathcal{X}_i = \vec{X}_i^{\mathrm{T}} \vec{X}_i$ defines $\vec{\mathcal{X}}$.

The term multiplying ω is always positive and hence is dissipative. Substituting $\vec{N}$ from Equation (10) gives an expression for the moments acting on the object as a function of position and rotational speed.

$$\tau = \underbrace{\mu W \vec{R}\mathbf{B}^{\mathrm{T}} \left(\mathbf{B}\mathbf{B}^{\mathrm{T}}\right)^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\tau_o} + \underbrace{\mu W \vec{R}\mathbf{B}^{\mathrm{T}} \left(\mathbf{B}\mathbf{B}^{\mathrm{T}}\right)^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\vec{k}_{s_\tau}} \vec{X}_{cm} + \vec{X}_{cm} \times \left(\vec{f}_o + \mathbf{k_s}\vec{X}_{cm}\right) - \mu\omega \left(\vec{\mathcal{X}}\vec{N} - W \vec{X}_{cm}^{T} \vec{X}_{cm}\right) \quad (16)$$

where $\mathbf{k_s}$ and $\vec{f}_o$ are the spring and offset constants from the translational dynamics, $\vec{k}_{s_\tau}$ is a 1×2 constant vector relating torque to position, and τ_o is a scalar constant torque. Note that nothing in the previous mathematics involved the orientation of the object, and hence, while the object rests on a particular set of supports, torque on the object is not a function of orientation. This is very important for determining stable orientations.

4 Design of Velocity Fields

A set of 9 constants quantifies the mass-spring-damper dynamics of the object as it rests on a single set of supports. A typical problem is to create a velocity field (described by $\mathbf{X}$ and $\mathbf{V}$) to produce mass-spring-damper behavior with uniform desired properties over the entire array. In particular, we specify an equilibrium position and return spring stiffnesses, and ensure rotational equilibrium at the translational equilibrium. The analysis relies on the following assumptions:

- The coordinate origin is at the desired equilibrium.
- The cells are arranged with mirror-symmetry around the coordinate axes.
- The object also has mirror symmetry.
- When the object rotates counterclockwise, more cells under the object lie in the first and third quadrants than in the second and fourth. More specifically, $\sum x_i y_i > 0$.

4.1 Translational Constants

Equation (14), specifies the translational dynamics of the object using 6 constants: 4 spring constants in $\mathbf{k_s}$ and two constant offset forces in $\vec{f}_o$. We consider only the case where $\mathbf{k_s}$ is a diagonal matrix, decoupling x and y motions of the object. The diagonal terms in $\mathbf{k_s}$ tend to pull the object towards a central equilibrium. The off-diagonal terms act as circulatory terms, moving the object around the equilibrium, and are not helpful for positioning the object. Eliminating the off-diagonal terms simplifies the design problem and improves the rotational properties of the field.

When the equilibrium position, $\vec{X}_{cm_e}$, and return spring strengths, $k_{s_{xx}}$ and $k_{s_{yy}}$, are specified, the object will move to the equilibrium position with the dynamics of the mass-spring-damper system shown in Figure 3. At equilibrium, $\vec{f}_o = -\mathbf{k_s}\vec{X}_{cm_e}$, so, in effect, $\vec{f}_o$ is specified. The design problem then becomes the problem of determining wheel velocities $\mathbf{V}$ given their positions (specified in $\mathbf{X}$ and equivalently in $\mathbf{B}$) and the constants $\mathbf{k_s}$ and $\vec{f}_o$. Section 3 defined the functional relationship from $\mathbf{V}$ to $\mathbf{k_s}$ and $\vec{f}_o$. This section describes a method to derive a suitable velocity matrix $\mathbf{V}$ from $\mathbf{k_s}$ and $\vec{f}_o$.

Equation 14 defines the constants $\mathbf{k_s}$ and $\vec{f}_o$ as

$$\begin{bmatrix} f_{o_x} & k_{s_{xx}} & k_{s_{xy}} \\ f_{o_y} & k_{s_{yx}} & k_{s_{yy}} \end{bmatrix} = \mu W \mathbf{V}\mathbf{B}^{\mathrm{T}} \left(\mathbf{B}\mathbf{B}^{\mathrm{T}}\right)^{-1}. \quad (17)$$

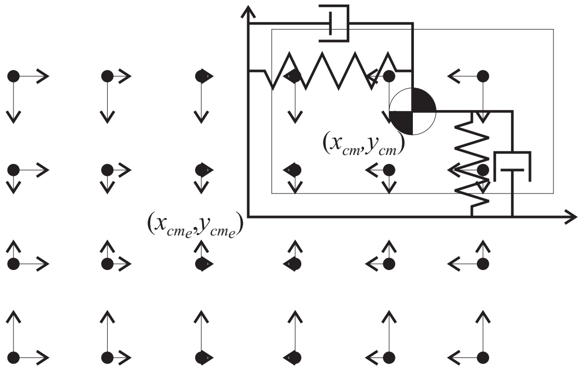

Figure 3: With $k_{s_{xy}} = k_{s_{yx}} = 0$ the object behaves like a mass-spring damper system.

We rewrite this equation in terms of the vector (of length $2n$) formed by stacking the transposes of the two rows of $\mathbf{V}$ ($\vec{V}_x^{\mathrm{T}}$ and $\vec{V}_y^{\mathrm{T}}$). Taking advantage of the symmetry of $\mathbf{BB}^{\mathrm{T}}$, the following relation holds.

$$\begin{bmatrix} f_{o_x} \\ k_{s_{xx}} \\ k_{s_{xy}} \\ f_{o_y} \\ k_{s_{yx}} \\ k_{s_{yy}} \end{bmatrix} = \mu W \underbrace{\left[\begin{array}{c|c} \left(\mathbf{BB}^{\mathrm{T}}\right)^{-1}\mathbf{B} & \mathbf{0} \\ \hline \mathbf{0} & \left(\mathbf{BB}^{\mathrm{T}}\right)^{-1}\mathbf{B} \end{array}\right]}_{\boldsymbol{\mathcal{B}}} \left[\begin{array}{c} \vec{V}_x^{\mathrm{T}} \\ \hline \vec{V}_y^{\mathrm{T}} \end{array}\right] \quad (18)$$

This produces a set of 6 equations and $2n$ unknowns.

Solve for the wheel speeds ($\vec{V}_x$ and $\vec{V}_y$) requires the inversion of the previous set of equations. This set of equations is underconstrained, so some freedom in the solution exists and further constraints are required. The *Penrose pseudo-inverse* accomplishes this by minimizing the sum of the squares of the wheel speeds. After applying properties of matrix transposes and inverses the pseudo-inverse, $\boldsymbol{\mathcal{B}}^{\dagger}$, is

$$\boldsymbol{\mathcal{B}}^{\dagger} \equiv \boldsymbol{\mathcal{B}}^{\mathrm{T}} \left(\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{B}}^{\mathrm{T}}\right)^{-1} = \left[\begin{array}{c|c} \mathbf{B}^{\mathrm{T}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{B}^{\mathrm{T}} \end{array}\right]. \quad (19)$$

Therefore, we can solve for the set of wheel speeds which will give an object the desired dynamics expressed by the desired equilibrium and spring constants (with $k_{s_{xy}} = k_{s_{yx}} = 0$ to decouple the x and y motions of the object).

$$\left[\begin{array}{c} \vec{V}_x^{\mathrm{T}} \\ \hline \vec{V}_y^{\mathrm{T}} \end{array}\right] = \frac{1}{\mu W} \left[\begin{array}{c|c} \mathbf{B}^{\mathrm{T}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{B}^{\mathrm{T}} \end{array}\right] \left[\begin{array}{c} f_{o_x} \\ k_{s_{xx}} \\ 0 \\ \hline f_{o_y} \\ 0 \\ k_{s_{yy}} \end{array}\right] \quad (20)$$

Since each row of $\mathbf{B}^{\mathrm{T}}$ is the vector $\left[1\ x_i\ y_i\right]$, each cell's wheel speeds are computed independently. In the diagonal $\mathbf{k_s}$ case, $f_{o_x} = -x_{cm_e} k_{s_{xx}}$, and $f_{o_y} = -y_{cm_e} k_{s_{yy}}$, so the wheel speeds are

$$\begin{aligned} V_{x_i} &= k_{s_{xx}} \left(x_i - x_{cm_e}\right), \text{ and} && (21) \\ V_{y_i} &= k_{s_{yy}} \left(y_i - y_{cm_e}\right), && (22) \end{aligned}$$

which is a field where the cells point towards the equilibrium (for negative $k_{s_{xx}}$ and $k_{s_{yy}}$), with velocities of each component proportional to the perpendicular distance to the corresponding axis (see Figure 3). Note that this is a discretized version of the continuous elliptic field described by Kavraki [3].

4.2 Rotational Constants

Section 3 showed that for an an object resting on a single set of supports, the torque is not a function of orientation. Therefore, it is not possible to construct a static velocity field which will orient an object more precisely than its range of motion which keeps it on a single set of supports. The black rectangle in Figure 4 demonstrates this free range of motion.

Locally, we can only assure that the object will be in rotational equilibrium when it is in translational equilibrium. Objects are oriented as they change support from one cell to the next. There are then three considerations for the object's orientation: (i) Torque is zero when translational force is zero (at $\vec{X}_{cm} = \vec{X}_{cm_e}$). ($ii$) When the object rotates about its equilibrium position, a change in supports induces a restoring torque. (iii) Given any starting position and equilibrium, the object will eventually reach the desired position and orientation. These considerations will be examined under the field derived in Section 4.1.

Without loss of generality, the origin of the coordinate system is placed at the desired equilibrium position. Therefore, when the object rotates about its translational equilibrium, $\vec{X}_{cm} = 0$ and $\vec{f}_o = 0$, and Equation 16 reduces to

$$\tau = \tau_o - \omega\mu\mathcal{X}\vec{N} \quad (23)$$

which is a constant applied torque with linear damping. We must then have $\tau_o = 0$ for the object to be in complete translational and rotational equilibrium. The expression for τ_o from Equation 16 is

$$\tau_o = \mu W \left[1\ 0\ 0\right] \left(\mathbf{BB}^{\mathrm{T}}\right)^{-1} \mathbf{B}\vec{R}^{\mathrm{T}}. \quad (24)$$

The vector $\vec{R}^{\mathrm{T}}$ can be expressed in terms of the stacked velocity vector. Furthermore, given $\vec{f}_o = 0$ (due to the choice of coordinate system) and $\mathbf{k_s}$ is diagonal (by design), the constant-torque-term is

$$\tau_o = \mu W \left[1\ 0\ 0\right] \left(\mathbf{B}\mathbf{B}^{\mathrm{T}}\right)^{-1} \mathbf{B} \cdot$$

$$\left[\begin{array}{ccc|ccc} -y_1 & & & x_1 & & \\ & \ddots & & & \ddots & \\ & & -y_n & & & x_n \end{array}\right] \left[\begin{array}{c|c} \mathbf{B}^{\mathrm{T}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{B}^{\mathrm{T}} \end{array}\right] \left[\begin{array}{c} 0 \\ k_{s_{xx}} \\ 0 \\ \hline 0 \\ 0 \\ k_{s_{yy}} \end{array}\right] \quad (25)$$

The rows of $\mathbf{B}$ are the ones vector, the vector of x positions, and the vector of y positions. Therefore, the terms produced by multiplying $\mathbf{B}$ with other vectors and matrices form sums of the x and y components of all the cell locations. For example,

$$\mathbf{B}\mathbf{B}^{\mathrm{T}} = \begin{bmatrix} n & \sum x_i & \sum y_i \\ \sum x_i & \sum {x_i}^2 & \sum x_i y_i \\ \sum y_i & \sum x_i y_i & \sum {y_i}^2 \end{bmatrix}. \quad (26)$$

In terms of these sums, Equation 25 becomes

$$\tau_o = \mu W \left[1\ 0\ 0\right] \cdot \begin{bmatrix} n & \sum x_i & \sum y_i \\ \sum x_i & \sum {x_i}^2 & \sum x_i y_i \\ \sum y_i & \sum x_i y_i & \sum {y_i}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum x_i y_i \\ \sum {x_i}^2 y_i \\ \sum x_i {y_i}^2 \end{bmatrix} (k_{s_{yy}} - k_{s_{xx}}) \quad (27)$$

In general, the torque resulting from these constants evaluated at the equilibrium position is not zero. However, consider the case where the cells on which an object rests are arranged *symmetrically* (mirrored in x and y) about the coordinate axes. Therefore, any term with odd powers of x_i or y_i in a summation will be identically zero. For example, in $\sum x_i y_i$, cells in the first and fourth quadrants cancel cells in the second and third quadrants, making $\sum x_i y_i = 0$. Similarly, $\sum {x_i}^2 y_i = 0$ and $\sum x_i {y_i}^2 = 0$. Therefore, the constant torque τ_o becomes identically zero such that the object will be in rotational equilibrium when resting on a mirror-symmetric set of supports at the translational equilibrium.

The symmetric arrangement of cells under the object depends on the object's shape and orientation. Figure 4 shows a rectangular object in three orientations at its equilibrium position with arrows at each cell indicating the magnitudes of the velocities at each cell. In this figure, we can see that both the symmetrically oriented object (solid black line) and the slightly perturbed object (dashed black line) have a symmetric set of forces, so do not feel a torque. However, the object which has rotated enough to change supports (dotted black line) has a set of supports which is symmetric about the origin (radially symmetric) rather than about the coordinate axes and will feel a torque.

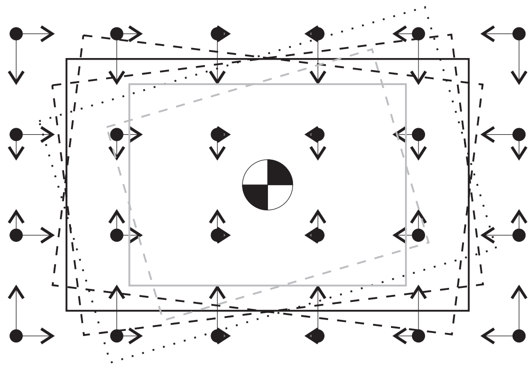

Figure 4: Rotation of two objects (black and grey) about their equilibrium positions. Black dashed lines represent the free (unorientable) range of motion of the black object. The grey object does not satisfy the "positive rotation" property. Note that $k_{s_{xx}}, k_{s_{yy}} < 0$ and $(k_{s_{yy}} - k_{s_{xx}}) < 0$.

Equation 27 shows that for a given arrangement of cells and a particular orientation of object, the direction of torque depends on the difference $k_{s_{yy}} - k_{s_{xx}}$. Therefore, these constants determine the stability of rotational equilibrium. If $k_{s_{xx}} = k_{s_{yy}}$ there will be no torque when the object rotates enough to shift cells.

If the object is mirror-symmetric itself, a stronger statement can be made about the direction of rotation. The resulting set of supports will be radially symmetric at any object orientation such that for every cell (x_i, y_i) there is a cell $(-x_i, -y_i)$. Thus, many of the terms in Equation 27 become zero.

$$\begin{aligned} \tau_o &= \mu W \left[1\ 0\ 0\right] \cdot \begin{bmatrix} n & 0 & 0 \\ 0 & \sum {x_i}^2 & \sum x_i y_i \\ 0 & \sum x_i y_i & \sum {y_i}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum x_i y_i \\ 0 \\ 0 \end{bmatrix} (k_{s_{yy}} - k_{s_{xx}}) \\ &= \frac{1}{n} \sum x_i y_i \left(k_{s_{yy}} - k_{s_{xx}}\right). \end{aligned} \quad (28)$$

A final assumption can be made that when the symmetric object rotates counterclockwise about the equilibrium position, $\sum x_i y_i > 0$. This is often true, since more of the object is in the first and third quadrants, and more first and third quadrant cells (giving $x_i y_i > 0$) are covered. The black rectangle in Figure 4 shows an object with this property. However, because of the discreteness of the array, some objects (for example, the grey rectangle in Figure 4) may have a negative $\sum x_i y_i$ for some counterclockwise rotations. A particular object can be checked for "orientability" on the array by analyzing it as it rotates about equilibrium and to see which cells it covers. For objects with this "positive rotation" property, there will be a restoring torque for $\left(k_{s_{yy}} - k_{s_{xx}}\right) < 0$ with a stable orientation.

Nothing has been said so far for torques on the object when its position is not at the equilibrium. To assure proper orientation, the object must first reach its equilibrium and then orient itself. Reaching the equilibrium is guaranteed regardless of orientation and cell distribution since the object's dynamics are that of a mass-spring-damper centered at the origin. Once translational equilibrium is reached, support changes due to rotation do not affect the translational dynamics.

Once position equilibrium is reached, a torque will be applied which tends to orient the object. For $\left(k_{s_{yy}} - k_{s_{xx}}\right) < 0$, a symmetric object satisfying the positive rotation property will rotate until it is aligned with the coordinate axes.

5 Conclusions

In this paper, a standard wheel velocity field was methodically designed to provide uniform, arbitrary spring constants and equilibrium position over the entire array regardless of which cells support the object. The resulting field, when only the non-circulatory spring constants are used, is an inward-pointing field with each wheel's velocity proportional to its the corresponding component of perpendicular distance to the equilibrium position.

This field was then analyzed for its rotational equilibrium properties. Interestingly, the torque on the object is not a function of the orientation because the supporting forces only depend on the position of the center of mass. Therefore, transitions from cell to cell as the object rotates about its equilibrium orient the object within the resolution limits of the discrete array. Given the assumption that the set of cells supporting the object at equilibrium is mirror-symmetric about the coordinate axes, it was found that a rotational equilibrium exists when the object is at translational equilibrium. A ramification of this assumption on a regular square-lattice array, such as the MDMS, is that the equilibrium position must be set either exactly on a cell or exactly midway between cells. Also, an object can be oriented only to angles of $\frac{n\pi}{4}$, for $n = 0, \ldots, 3$.

The torque developed by the inward-pointing field was analyzed as the object rotates about equilibrium. It was determined that, as long as the two spring constants are not equal, when the object rotates and the supports change, there will be a restoring torque on a symmetric object with the property that $\sum x_i y_i > 0$ for all the cells under the object when the object rotates counterclockwise. This positive rotation property is an artifact of the discrete array and holds for many object sizes and shapes. Furthermore, in the the limit of many cells spaced closely together (e.g., a continuous array), this property always holds for symmetric objects, as implied by Kavraki. The discrete case requires a simple analysis to check if an object meets this assumption.

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