Small Slope Implies Low Speed In Passive Dynamic Walking

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Abstract

In this paper we consider the passive dynamic walking machines originally conceived and studied by McGeer (1990a). For passive walking on arbitrarily small slopes, we show that any existing gaits must be correspondingly slow. First, we describe a walking motion, and restrict our attention to passive machines that follow this basic prescription. We simplify the description of arbitrary mass distributions using equivalent two-point-mass distributions. We present the idea of a "local mass matrix" **M** that is associated with impulse-momentum and energy calculations during collisions. We consider physically-meaningful limiting cases of mass distributions for which **M** is singular. We present two different lines of argument for non-singular and singular **M**, showing that steady walking motions on vanishingly small slopes must have vanishingly small step lengths. Finally, we argue that under the action of passive, pendulum-like free-swinging dynamics, walking with vanishingly small step lengths must occur with vanishingly small speed. This concludes the proof. The result in this paper provides constraints on the mass distribution of passive walking machines that will walk at arbitrarily small slopes, as explained in a separate paper (Garcia et al. (1998b)).

1 Introduction

Passive machines that walk on shallow slopes were first studied by McGeer (1990a, 1990b), inspired by the freely-swinging double- and triple-pendulum models of a leg proposed by Mochon and McMahon (1980). These machines consist of hinged rigid bodies that make collisional and rolling contact with a rigid sloped surface. Passive bipedal walking machines can be *straight-legged* or have hinged knees (*kneed walkers*).

1.1 Description Of Walking

Walking machines and their gaits are described in detail in other papers (e.g., McGeer (1990a, 1990b) and Garcia et al. (1998b)). The rest of this section appears almost verbatim in the companion paper of Garcia et al. (1998b), but we include it here for completeness. Readers familiar with McGeer's walking machines may skip the rest of this subsection.

The type of machine we study here, a copy of McGeer's basic design, is shown schematically in fig. 1 (from Garcia et al. (1998b)). It consists of a *swing* leg (not in contact with the ground) and a *stance* leg (touching the ground), connected by a frictionless hinge at the *hip*. Each leg has a rigid thigh and shank. The stance knee is locked. For kneed walkers, the swing knee is a frictionless hinge with a knee-stop preventing hyperextension. The two legs are identical.

A steady walking motion or a *gait cycle* proceeds as follows.

At the start of the cycle, both legs are straight and touch the ground. This instant is just after *heelstrike*. The velocities at this instant are, of course, nonzero. Then the trailing leg, or swing leg, (fig. 1) lifts off the ground and swings forward, and the swing knee flexes. The leading leg maintains contact with the ground, remains straight, and rolls forward. This is the stance leg. *Kneestrike* is assumed to be instantaneous; the swing knee locks with no rebound and the swing leg remains straight thereafter. The mechanism thus switches from a *three-link* to a *two-link* kinematic chain. At the end of the cycle, the swing foot has a collision with the ground at *heelstrike* (instantaneous and plastic). At the instant of double-support (i.e., contact at both feet), the legs exchange stance and swing roles, and a collisional impulse acts at the new contact point on the new stance foot.

Straight-legged walkers have locked knees and are therefore always in two-link mode between heelstrikes.

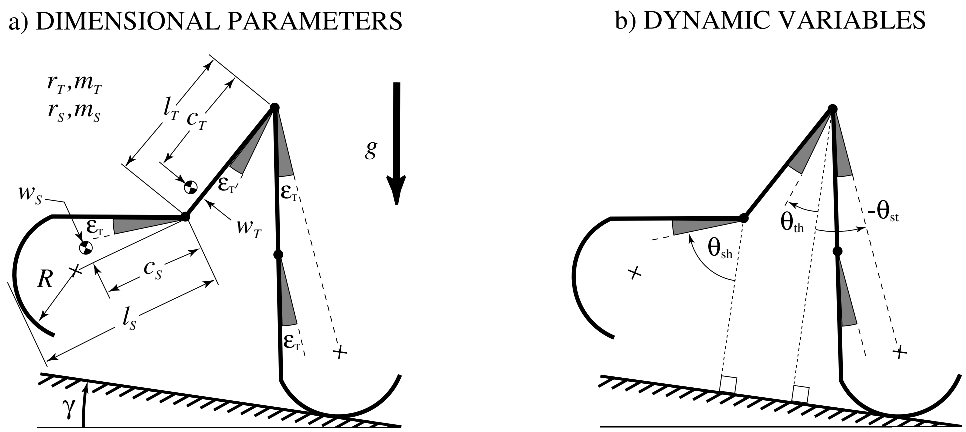

Figure 1: McGeer's kneed walking model. Shown are (a) model parameters, and (b) dynamic variables. Radii of gyration and masses of thigh and shank are denoted by r_t, m_t, r_s, and m_s, respectively. The foot is a circular arc centered at the "+". Dynamic variable values θ_{st}, θ_{th}, and θ_{sh} are measured from ground-normal to lines offset by ϵ_T from their respective segments. A stop (not shown) at each knee prevents hyperextension of either knee. In straight-legged models, the knee is locked. Detailed simulation results may be found in a companion paper (Garcia et. al. (1998b)).

1.2 Outline And Motivation Of This Work

In this paper, we prove that for McGeer-like two-dimensional passive walking machines, if steady walking motions exist at vanishingly small slopes, then they must be at vanishingly small speeds. The reason for this is that plastic (no-slip, no-bounce) collisions between foot and ground dissipate energy at nonzero speeds.

Intuitively, this may initially appear trivial. However, it is not *a priori* obvious that (say) a kneed walker could not possibly have passive gaits with gentle, non-dissipative foot placements. Controlled walkers can certainly have gaits with such gentle foot placements (Blajer and Schiehlen (1992)). As argued by Winter (1987), gentle foot placements during walking may also be a human strategy to minimize energy loss and body stress at heelstrike. Moreover, it is known that running machines with springs on the feet can have dissipation-free running motions. Also, there are models of brachiation from Ruina (1997) that allow dissipation-free support transfer under reasonable idealizations. So it seems worthwhile to prove that for McGeer-like machines, small slopes do imply slow speeds. This result is of interest in the study of walking dynamics because it guides designs of highly efficient walking machines that walk at very small slopes, and also provides basic limitations on the efficiency of such walking machines, as discussed in detail in a companion paper (Garcia et al. (1998b)).

In this paper, we consider machines that follow the prescription of walking given above. Thus, our proof does not apply to machines with other types of motions with intermittent contact, such as (say) a machine with springs on the feet that might allow dissipation-free support transfer; or that has springs at the knees that let it possibly reach a gentle, dissipation-free heelstrike at a *bent-knee* configuration or three-link mode without falling down; or where the normal force from the ground at the trailing leg drops to zero at some point, letting the machine rise into the air for a portion of the gait cycle; or a machine with three or more legs that allow gentle, dissipation-free transfer of support between legs, and so on.

We also assume that the foot radius R of the walking machines is smaller than the leg length l. McGeer (1989) also studied "silly wheels" with $R = l$, for which non-dissipative walking motions are possible on perfectly flat ground.

The rest of the paper is structured as follows. First, we present the basic impulse-momentum and energy loss calculations for *heelstrike*, for both straight-legged and kneed walkers. This discussion introduces a 2×2 matrix $\mathbf{M}$ which is symmetric and positive semi-definite. Next, we examine the conditions under which $\mathbf{M}$ is strictly positive definite, and the conditions under which it is singular, again for both straight-legged and kneed walkers. Depending on whether $\mathbf{M}$ is singular or not, we then present two different lines of argument showing that walking solutions on vanishingly small slopes must be with vanishingly small *step lengths*, again for both straight-legged and kneed walkers. Finally, we discuss why walking with vanishingly small step lengths must necessarily be at vanishingly small speeds, for these passive, gravity-powered machines. That concludes our proof.

2 Impulse-Momentum And Energy Calculations In Collisions

Consider a linkage consisting of several effectively rigid objects connected by ideal, frictionless bearings, as shown schematically in fig. 2. The linkage makes collisional contact with a rigid, immovable floor at a single point C. We assume that at the instant of collision, an impulse vector $\mathbf{P}$ acts at point C. Let the pre-collision velocity of point C be the vector $\mathbf{v}_i$, and the post-collision

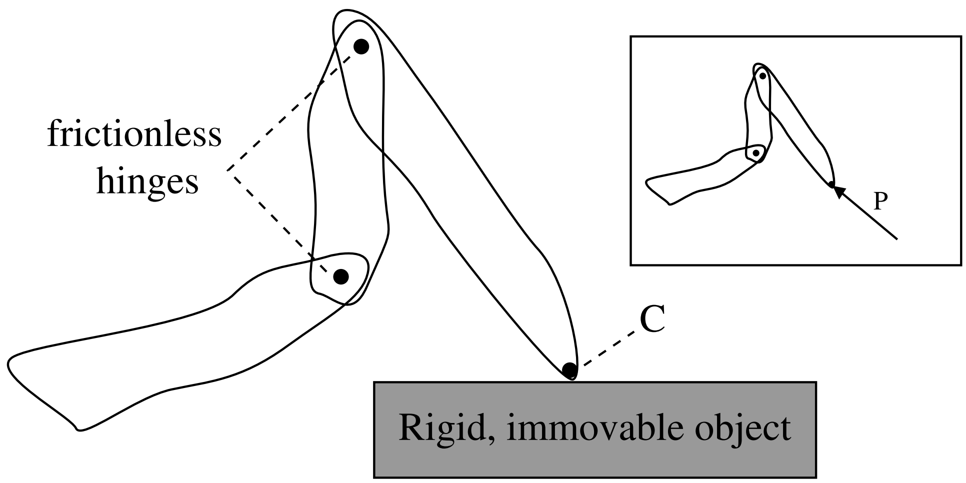

Figure 2: A free-floating linkage collides with a rigid immovable object (floor).

velocity of C be $\mathbf{v}_f$.

Using standard rigid body dynamics, one can show that the *change* in the velocity of point C, i.e., the vector $\mathbf{v}_f - \mathbf{v}_i$, is linearly related to the impulse vector $\mathbf{P}$ by a symmetric, positive definite tensor $\mathbf{M}$, i.e., the collisional interaction can be described by an equation of the form

$$\mathbf{P} = \mathbf{M} \cdot (\mathbf{v}_f - \mathbf{v}_i) \tag{1}$$

The tensor $\mathbf{M}$ depends on the masses, moments of inertia, positions of pivot points, and the instantaneous configuration of the linkage. Assuming that a coordinate system (along normal and tangential directions at the contact surface) has been chosen and that we are considering 2-D systems, the tensor $\mathbf{M}$ is represented by a 2×2 matrix of components, which (with some abuse of notation) we also call $\mathbf{M}$.

It is also well known (see e.g., Chatterjee (1997) and Smith (1991)) that the kinetic energy dissipated in the collision is equal to

$$\frac{1}{2}\mathbf{v}_i \cdot \mathbf{M} \cdot \mathbf{v}_i - \frac{1}{2}\mathbf{v}_f \cdot \mathbf{M} \cdot \mathbf{v}_f$$

which, in the special case of a perfectly plastic (no-slip, no-bounce) collision where $\mathbf{v}_f = 0$, reduces to

$$\text{Energy loss } = \frac{1}{2}\mathbf{v}_i \cdot \mathbf{M} \cdot \mathbf{v}_i\,. \tag{2}$$

Also, for such a plastic collision, we have the impulse

$$\mathbf{P} = -\mathbf{M} \cdot \mathbf{v}_i\,. \tag{3}$$

Note that if the mass distribution of the contacting rigid body (the first one in the chain) is singular in some way, then the tensor $\mathbf{M}$ can be singular, i.e., symmetric and positive semi-definite. For example, the particularly simple walker considered in Garcia et al. (1998a) has a singular mass distribution. Therefore, we next consider the ways in which $\mathbf{M}$ can be singular.

3 Singular Mass Distribution In Walking Machines

3.1 Two Point Mass Representation Of An Arbitrary Rigid Body

As seen in fig. 1, the walker is described by several system parameters including masses, center of mass positions, and moments of inertia. In order to simplify the discussion, we use a *dynamically equivalent* description where the mass distribution consists of a few point masses only.

In two dimensions with constant gravity, any single rigid body with total mass M, center of mass a distance $r > 0$ from an arbitrarily chosen reference point R, and moment of inertia $J > 0$ about its center of mass is dynamically equivalent to a pair of point masses, one of which, say m, is at R and the other $(M - m)$ is a distance s from R, where m and s are to be chosen appropriately.

To see this, observe that the total mass is the same, $m + (M - m) = M$; center of mass position and the moment of inertia will also be the same if m and s satisfy the simultaneous equations:

$$Mr = (M - m)s, \qquad \text{and} \qquad J + Mr^2 = (M - m)s^2, \tag{4}$$

where point R, the center of mass, and the mass $M - m$ are collinear. Equation 4 may be solved to obtain

$$m = \frac{J}{J + Mr^2}\, M, \qquad \text{and} \qquad s = \frac{Mr^2 + J}{Mr^2}\, r\,. \tag{5}$$

Thus for 2-D straight-leg walking machines, treating each leg as having two point masses, one (say) at the hip and one at some other, arbitrary point, involves no lack of generality (the center of mass cannot be exactly on the hip, for swinging motions of the swing leg to be possible).

To emphasize the arbitrariness (or generality) with which the locations of the point masses may be chosen, consider the example depicted in fig. 3. A rigid body, with total mass M and moment of inertia J about center of mass O, may be represented using two appropriate point masses at points A and C, where the masses at A and C as well as the distance OC are determined by M, J and the location of point A. The same body may, if and when convenient, also be represented

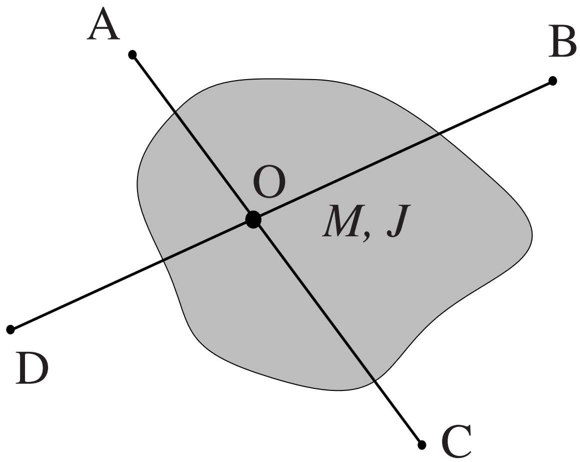

Figure 3: Two possible locations for two-point-mass representations of a single rigid body.

using two other point masses, at (say) B and D, with the masses at B and D as well as the distance OD being determined by M, J and the location of point B.

3.2 Invertibility Test For Matrix M

It is well known (see e.g., Chatterjee (1997)) that the matrix **M** provides a linear relation between any force **F** at the contact point C (fig. 2) and the resulting acceleration of point C, $\mathbf{a}_{\mathrm{C}}$, provided all bounded external forces and torques, forces and torques from springs at joints (if any), as well as centripetal terms (or "ω^2 terms") are neglected:

$$\mathbf{F} = \mathbf{M} \cdot \mathbf{a}_{\mathrm{C}} \,. \tag{6}$$

In other words, if we imagine the system to start from rest (hence, no ω^2 terms), ignore the effects of springs at joints (if any), and imagine that there are no forces acting on it except a single force **F** acting on point C, then the matrix **M** linearly relates the acceleration of point C to the force **F**.

In our analysis, we call a walker singular if the matrix **M** for its heelstrike collision calculation is singular. Otherwise, we will call the walker non-singular.

We will study singular **M** using Eq. 6, and for systems where the mass distribution of each leg is represented using a pair of suitably placed point masses, as described in subsection 3.1.

We will use the following simple test for invertibility of **M**:

Test for Invertibility: Consider a linkage (as in fig. 2) at rest. Let the mass distributions of each component piece be represented as two point masses at two distinct locations. One or more of these point masses may be zero. If it is possible to move the point C, obeying the constraints at the hinges, and without moving any of the *nonzero* point masses, then **M** is singular. Otherwise, **M** is invertible.

The basis for the test is this. If the point C can move without setting any nonzero mass in motion, then it can be moved with zero force (since there is no friction in the linkage, and no forces on the system including gravity forces, the only resistance is inertial). By Eq. 6, **M** must then be singular. On the other hand, if moving C means moving some nonzero mass, then no nonzero force can do it, so **M** is invertible.

3.3 Physically Meaningful Singular Mass Distributions

Although a study of singular walkers is useful (e.g., Garcia et al. (1998a)), we restrict the class of singular distributions we will consider. For walkers with singular mass distributions, we assume that the limiting dynamic behavior of the walker is "physically meaningful," in that a single leg of the walker, when suspended from the hip under gravity, must have pendulum-like oscillation frequencies which are strictly bounded even though the mass distribution is singular.

We remark that a basic feature of the passive walking motions we are interested in is pendulum-like motions of the swing leg that occur on time scales comparable to the step period, so the physical meaningfulness criterion given here is hardly restrictive.

3.4 Singular Mass Distributions For Straight-Legged Walkers

Consider the straight-legged walker shown schematically in fig. 4, with a nonzero step length. [1] Note that the stick-figure drawing is not necessarily representative of the actual shape of the legs. The drawing is meant to emphasize the single-rigid-body legs, circular feet, and positions of point masses. The hip is at H, and the second point masses on the two legs are at R and S. Since the two legs are identical and the step length is nonzero, the points R and S do not coincide. The total

[1] We explain here why the foot radius should be strictly smaller than the leg length. For foot radius equal to leg length, the system is singular in that the ground-contact points for the two feet, in fig. 4, coincide no matter what the angle between the legs. If the foot radius is larger than the leg length, then the contact point of the trailing foot is ahead of the walker while the contact point of the leading foot is behind the walker. Such a configuration is reminiscent of walking while wearing skis, and not representative of anthropomorphic bipedal walking.

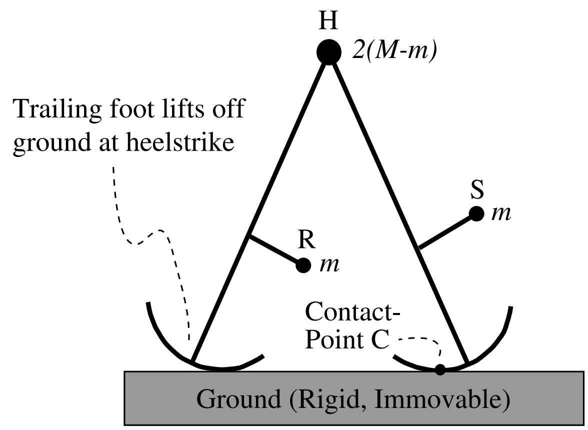

Figure 4: A straight-legged walker with the mass distribution of each leg expressed as the equivalent pair of point masses, with one point mass at the hip.

mass of each leg is taken to be M. The point masses at R and S are m each. The two point masses $M-m$ at the hip join up to make one point mass, $2(M-m)$.

This is a two-link version of fig. 2. The contact between the ground and the trailing foot is ignored, since by assumption there is no impulse at that contact during heelstrike.

By the discussion in subsection 3.2, we conclude the following.

If neither m nor $M-m$ is negligible, then $\mathbf{M}$ is invertible.

If m is negligible while $M-m$ is not, then it is possible to rotate the stance leg about the hip H, without moving any nonzero mass. Thus, for this case, $\mathbf{M}$ is singular.

If $M-m$ is negligible while m is not, then $\mathbf{M}$ is invertible because it is impossible to move C without moving either R or S.

The case of both m as well as $M-m$ being negligible is not meaningful for mechanisms powered only by gravity. Also, we do not consider the singular case where the distances HR and HS become arbitrarily small (not physically meaningful as described earlier).

Therefore, we conclude that for straight-legged walking, the only singular mass distribution of interest is one where the entire mass of the walker is essentially concentrated at the hip (as, e.g., in the "simplest" walker of Garcia et al. (1998a)).

3.5 Singular Mass Distributions For Kneed Walkers

We now consider the possible cases of singular mass distribution in kneed walking. Since the walker is in two-link mode just prior to heelstrike, we first examine the effective, instantaneous mass distribution for this straight-leg configuration, as shown in fig. 4.

As before, the total mass of each leg is assumed to be M. If m is negligible compared to $M-m$, then $\mathbf{M}$ is singular as discussed above, because C can be moved without moving H.

Also as discussed above, the case of both m and $M-m$ being negligible is not relevant for gravity-powered machines.

Now we examine the case of $M-m \ll m$. That is, in the two-point-mass representation of the straightened leg with one point mass at the hip, we consider the case where the hip mass is negligible and essentially all the mass is concentrated at the other point, R.

For straight-legged walkers, we know this case is non-singular. For kneed walkers, we must account for the possible bending of the trailing leg (i.e., the old stance leg or the new swing leg) as a result of the collision impulse. We still treat the new stance leg as a single rigid object, because by assumption it stays straight after heelstrike (see the Introduction).

Figure 5 shows just the trailing leg of the walker, along with the position of the knee. The "actual" geometry is shown with heavy black lines, while the underlying gray shows the sketch from fig. 4. In other words, the knee is allowed here to be off the line connecting the hip with the center of the foot's circle.

If the leg is allowed to bend, we must consider the mass distributions of the shank and thigh separately.

We now express the mass distribution of the shank using two equivalent point masses, but this time with one mass m_{sh} at point R of fig. 4, and the other at some arbitrary point (which need not be the hip). [2] Similarly, we express the mass distribution of the thigh using two equivalent point masses, with a point mass m_{th} at point R and another point mass at some arbitrary point.

We remark that the positions of these two new point masses for the shank and thigh are not independent of each other. Their exact positions will turn out to be irrelevant, however, and so we avoid calculating the exact constraints on these positions. The schematic diagram (fig. 5) is not meant to reflect these constraints, but merely convey the idea that the mass distributions of the thigh and shank are each represented by two point masses, one at the point R and another at "some other" point.

[2] Note that m_{sh} is not assumed to be equal to the shank's total mass m_S (of fig. 1). The two happen to be equal if the hip mass $M-m$ is negligible.

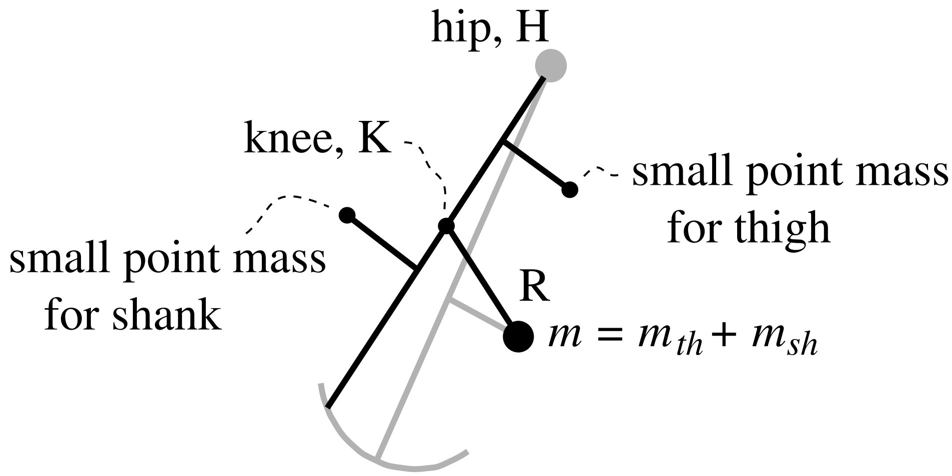

Figure 5: The trailing leg of a kneed walker with singular mass distribution, redrawn to show the knee.

Now recall that we are only considering $m \gg M - m > 0$ (or, $m \to M$ from below) in fig. 4. Thus, the moment of inertia of the entire leg about R, were the knees to be artificially locked, is negligible compared to Ml^2 (where l is the leg length). This implies, in the singular limit, that $m = m_{th} + m_{sh}$, as indicated in fig. 5. There are no finite (nonzero) masses anywhere else other than at the point R. [3]

We now apply the test for invertibility. First, if m_{th} is not negligible, then **M** is invertible due to the reasons discussed for the straight-legged walker: it is not possible to move C without moving some nonzero point mass either on the new stance leg, or on the trailing thigh.

Next, we examine the singular limit of the case where $m_{th} \ll m \approx m_{sh}$, i.e., when m_{th} is negligible. In this case, there are two possibilities: either R coincides with K, or it does not.

If R happens to be exactly on the knee K, then the point mass m_{sh} might as well be attached to the thigh instead of to the shank as far as the dynamics is concerned, and so again **M** is invertible.

Finally, if $m_{th} \ll m \approx m_{sh}$, and R does not coincide with K, then the system is not physically meaningful in the sense described above, because the double-pendulum-like oscillations of the leg, if suspended from the hip and with the knee constraint removed, would have one natural frequency that was infinite in the singular limit.

Therefore, we conclude that the entire mass of a singular *kneed* walker, when it is in the straight-

[3] As a result, in the singular limit, m_{th} and m_{sh} of fig. 5 must be equal to m_T and m_S of fig. 1. This observation is not necessary for the proof.

leg configuration, is essentially concentrated at the hip, just as in the case of a straight-legged walker.

3.6 Rank Of Singular Matrix M For Singular Walkers

As discussed above, $\mathbf{M}$ is a 2×2 matrix. If it is singular, then its rank must be one, and not zero.

That the rank cannot be zero follows from the observation that even for the singular case there must be a finite (nonzero) mass at the hip H (as discussed above). If there is a nonzero mass at the hip in fig. 4, then in moving the point C *towards* the hip H we must move some nonzero mass, for both straight-legged and kneed walkers. It follows that for moving C in that direction, some nonzero force is required. By Eq. 6, then, $\mathbf{M}$ cannot be identically zero; the rank of $\mathbf{M}$ is always at least one.

4 Small Slope Implies Small Steps

We now show that steady passive walking of McGeer-like machines in 2-D on infinitesimally small slopes must necessarily occur with infinitesimally small steps, if it occurs at all. As mentioned earlier, we assume $R < l$.

The proof is based on contradiction. We first assume that walking is possible with strictly nonzero steps on arbitrarily small slopes. This allows us to consider the limiting case of exactly zero slope, on which the machine (by assumption) can walk with nonzero step size. On zero slope, zero energy must be dissipated in each walking step, including heelstrike. We show that some energy must be dissipated, however, for walking motions to exist. This provides a contradiction.

The proof is different for singular and non-singular mass distributions. For each case, a proof is first presented for straight legged walkers, and then it is discussed why the proof holds for kneed walkers also.

4.1 Proof For Singular Walkers

For singular straight-legged walkers, the entire mass of the walker is essentially concentrated at the hip, as discussed earlier.

For simplicity, we consider two cases.

First, at the instant of heelstrike, if the hip H has exactly zero velocity, then the old stance leg must be stationary. The old swing leg must then be rotating about the hip. Due to the plastic collision, the old swing leg comes to a complete stop, and the hip is still at rest. Now the whole walker is at rest. By inspection of fig. 4, it is clear that this position is one of equilibrium. Walking cannot continue in this case.

Second, consider the case where heelstrike is reached with H having a nonzero velocity. Since the stance leg rolls on the ground, hip H has a velocity in a direction that has a strictly nonzero component along the line HC (fig. 4) for nonzero step length. Since, after heelstrike, the velocity of the hip must be perpendicular to HC, the velocity of H must change at heelstrike. Thus, there must be a nonzero collision impulse. This means that (Eq. 3) $\mathbf{M} \cdot \mathbf{v}_i \neq \mathbf{0}$, and so (Eq. 2) some energy is dissipated. (Contradiction.)

For singular kneed walkers, recall that some time prior to heelstrike the walker enters two-link mode (see the Introduction). Also, as discussed earlier, essentially the entire mass in the *straight-leg* configuration is concentrated at the hip. Half of this mass belongs to the leading or new stance leg, which remains straight. Therefore the velocity of some nonzero mass changes direction. The rest of the argument is the same as for straight-legged walkers.

4.2 Proof For Nonsingular Walkers

For non-singular walkers, the mass distribution can be complicated, and the dynamics is not as simple as for singular walkers. On the other hand, any heelstrike collision with nonzero foot velocity must dissipate energy, precluding a walking motion. Thus, the foot collision *must* be gentle, i.e., the foot must come to rest on the ground with zero velocity. With this in mind, we examine a small portion of the whole solution, close to the instant of heelstrike, using a suitable coordinate system that is well-defined in that neighborhood.

We begin with a straight-legged walker. To obtain a contradiction, we assume that a gait cycle exists with nonzero step length on flat ground.

As long as rolling contact at the trailing foot is maintained, the system has two degrees of freedom. As shown in figure 6, we use the x and y coordinates of the point of impending contact on the foot, measured from the corresponding point on the ground, as generalized coordinates that describe the system. Note that for nonzero step length, the coordinates x and y are well-defined.

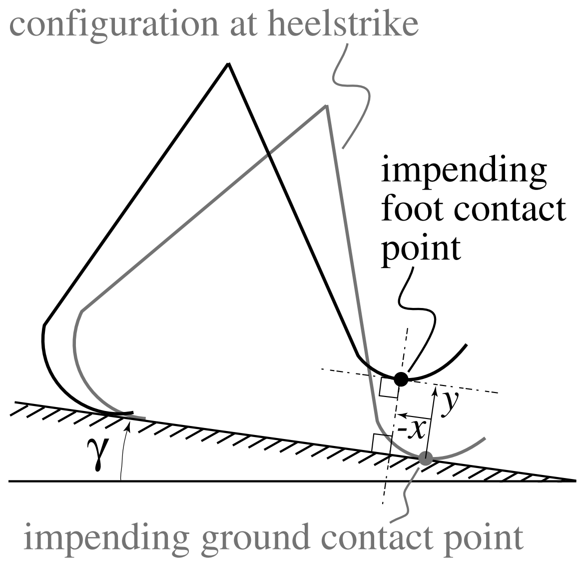

Figure 6: Close to heelstrike, the x and y coordinates of any point on the foot relative to any point on the ground can be used as generalized coordinates to describe the configuration of the walker. In particular, given that a gait cycle exists, these points can be chosen to be the points at which contact will occur at heelstrike.

Heelstrike occurs when $x = y = 0$. By assumption, at that instant, $\dot{x} = \dot{y} = 0$ also.[4]

Using $\mathbf{q} = \{x, y\}^T$ as a coordinate vector, we can write equations of motion for the walker that will be valid in some neighborhood of $\mathbf{q} = \mathbf{0}$ (ignoring the new contact with the ground that might occur at heelstrike, and assuming that contact at the stance-foot or trailing foot continues).

Note also that the system is conservative with scleronomic (time-independent) constraints. Its Lagrangian will therefore be of the form

$$L = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{A}(\mathbf{q})\dot{\mathbf{q}} - V(\mathbf{q}),$$

where $V(\mathbf{q})$ represents the potential energy of the system, and $\mathbf{A}(\mathbf{q})$ is a configuration-dependent, symmetric, positive definite matrix. If the system were dissipative, passive walking motions could not occur on flat ground.

[4]Note that for exactly zero step length, the foot circles coincide. In this configuration, the coordinates x and y are not independent. This may be easily seen because the hip is directly above the foot contact C. Starting from this configuration, keeping $x = 0$, y cannot be given an infinitesimal positive increment since it would lift both feet off the ground, violating the contact constraint for the stance foot.

We write the equations of motion,

$$\mathbf{A}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{f}(\mathbf{q}) + v.d.t. = \mathbf{0}, \tag{7}$$

where $\mathbf{f}$ represents potential forces acting on the system, and $v.d.t.$ stands for "velocity dependent terms," which only include terms that are of degree two in $\dot{x}$ and $\dot{y}$ (i.e., quadratic terms). A simple property of such systems is symmetry with respect to time reversal: changing time t to $-t$ leaves the equations of motion unchanged.

Since the system is autonomous, we let the instant of heelstrike occur at $t = 0$. At $t = 0$, we have $\mathbf{q} = \mathbf{0}$ and $\dot{\mathbf{q}} = \mathbf{0}$. These initial conditions are unchanged if we change t to $-t$. Assuming no contact with the ground, Eq. 7 continues to hold. If we define $\mathbf{q}_1(t) = \mathbf{q}(-t)$, then $\mathbf{q}_1(t)$ satisfies the equations of motion, as well as the initial conditions. Therefore, the walker exactly retraces its path, i.e., the leading foot lifts off the ground.

Recall that a walking motion requires the leading foot to stay in contact with the ground, while the trailing foot lifts off (since we assume plastic collisions at heelstrike). Thus a walking solution cannot occur without energy dissipation (a contradiction).

Therefore straight-legged walkers with round feet (or point-feet, as a special case), and non-singular mass distributions, cannot walk passively on near-zero slopes with finite (nonzero) step length.

For kneed walkers, again, we note that at some instant prior to heelstrike, the system switches to two-link mode. Thus, the arguments given above apply to kneed walkers also.

5 Small Steps Imply Small Speed

As shown above, for walking on a small slope γ, the step length (say S) must be also small.

We now show that if the step length is small, the overall walking speed must be small also. The basic reasoning is that the walking motion arises out of a combination of a "falling" stance leg and a swinging leg, under the action of gravity. The time scale of these interactions cannot become arbitrarily fast. Hence, smaller and smaller steps are taken on essentially the same time scale. (In fact, for the point-foot model of Garcia et al. (1998a), the step period τ is found to be $\tau = \tau_0 + \tau_1\gamma^{2/3} + \ldots$, showing how $\tau \to \tau_0$ as $\gamma \to 0$.) Thus, as the slope goes to zero, the speed (say S/τ) goes to zero, too.

5.1 Nonsingular Walkers

A somewhat more formal argument is now presented for walkers with non-singular mass distributions. This argument applies to both straight-legged and kneed walkers.

As the slope γ approaches zero, let the step length be $\epsilon(\gamma) \ll 1$. If the walking speed is $\mathcal{O}(1)$, i.e., not "small," then the time period of each step must be $\mathcal{O}(\epsilon)$ also. The gravitational energy dissipated per walking step must be $\mathcal{O}(\epsilon\gamma)$ (step length $\times$ slope), therefore the velocity of the heel at heelstrike must be asymptotically much smaller than one, i.e., $o(1)$. Since the average velocity of the heel is the same as for the whole walker, there must be an instant in the walking cycle when the velocity of the heel is $\mathcal{O}(1)$.

This means that within the time period of one step, i.e., $\mathcal{O}(\epsilon)$, the change in the velocity of the heel without any collisional effects is at least $\mathcal{O}(1)$. Thus, the acceleration of the heel needs to be at least $\mathcal{O}(1/\epsilon)$, i.e., unbounded for walking at near-zero slopes. Since this cannot occur for a passive system acted upon only by gravity, the walking speed cannot be $\mathcal{O}(1)$ and must be near-zero also.

5.2 Singular Walkers

For singular walkers we cannot conclude from simple energy calculations that the heel velocity at heelstrike is necessarily small, since $\mathbf{M}$ is singular.

5.2.1 Singular Straight-Legged Walkers

We first consider straight-legged singular walkers. As discussed earlier, the only singular mass distribution of interest is the case when all the mass is essentially concentrated at the hip. In this case the stance leg dynamics is that of an inverted pendulum $\mathcal{O}(\epsilon)$ close to vertical, so the maximum acceleration of the hip is also $\mathcal{O}(\epsilon)$. Over a step period of $\mathcal{O}(\epsilon)$ (by assumption), the fluctuation in the hip velocity is then $\mathcal{O}(\epsilon^2)$. Consequently, collision-induced velocity changes are also $\mathcal{O}(\epsilon^2)$.

The swing leg dynamics for singular walkers is that of a pendulum with a moving support (the hip), but now the hip moves at roughly constant velocity. Let us look at the system in a Newtonian frame which translates at the mean velocity of the hip. In this frame of reference the swing leg is a pendulum with an "almost" fixed support. Velocity changes of the swing leg caused by collision-induced discontinuities in the hip velocity are $\mathcal{O}(\epsilon^2)$, as are accelerations caused by accelerations of the support. Over times of $\mathcal{O}(\epsilon)$, i.e., the step period, these can cause displacements of at most

$\mathcal{O}(\epsilon^3)$. However, since the step length is $\mathcal{O}(\epsilon)$, in the moving frame the pendulum roughly executes oscillations of amplitude $\mathcal{O}(\epsilon)$ in $\mathcal{O}(\epsilon)$ time. Thus, ignoring the higher order effects, we see that the natural frequency of gravity-induced oscillations of the pendulum must be $\mathcal{O}(1/\epsilon)$. Since this is not possible as $\epsilon \to 0$, the walker speed cannot be $\mathcal{O}(1)$, and must be near-zero also.

5.2.2 Singular Kneed Walkers

In extending the above argument to kneed singular walkers, we must consider the fact that the walker is in three-link mode for a certain portion of the gait cycle. Recall that in the straight-leg configuration of a kneed walker, the mass is essentially concentrated at the hip. However, some of this mass belongs to the thigh and the rest belongs to the shank. The portion that belongs to the shank will shift away from the hip when the walker is in three-link mode. This situation is illustrated in fig. 7.

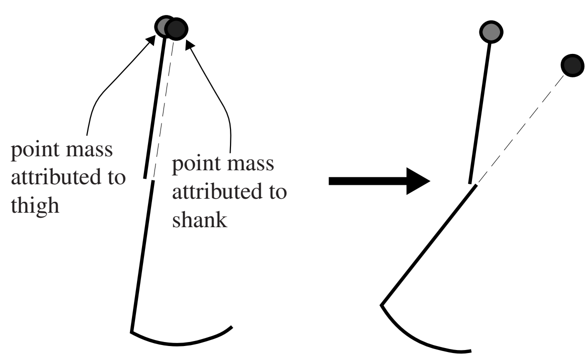

Figure 7: The (old) stance leg of a kneed singular walker may flex after heelstrike. The total mass of the straightened leg, being concentrated at the hip, may be viewed as two coincident point masses (shown artificially separated for clarity), one belonging to the thigh and the other to the shank. If the knee flexes, these two point masses do not coincide any more.

Shank mass negligible: If the shank mass is negligible compared to the thigh mass, then the previous argument for straight-legged walkers goes through unchanged except for the fact that the swing leg is a double pendulum for a portion of the gait cycle.

Shank Mass Not Negligible: We also consider the case when the shank mass is not negligible. In this case, we observe that the shank rotates by angles of $\mathcal{O}(\epsilon)$ relative to the thigh, i.e., the walking motion does not involve wild swings of the shank through angles much larger than the other angles involved in the motion. [5] In this case, however, the system is always $\mathcal{O}(\epsilon)$ close to the straight-leg configuration. As such, the acceleration of the hip is still $\mathcal{O}(\epsilon)$, and the previous argument goes through again.

We remark that singular kneed walkers with significant shank mass are non-anthropomorphic. Recall that for singular kneed walkers in a straight leg configuration, the entire mass must essentially be concentrated at the hip. Thus, if the shank mass is significant, then the center of mass of the shank must be at the hip. The shank, vertically hanging from the knee, is in unstable static equilibrium. The shank will tend to swing into an upside-down position. However, this can presumably be rectified with a torsion spring at the knee, giving a still-passive device that is allowed by our proof. So we have allowed this non-anthropomorphic mass distribution in our discussion of passive walking machines.

This concludes our proof of the claim made at the beginning of this paper: passive dynamic walking machines in 2-D, that follow the basic description of walking motions given in the introduction, cannot have fast walking motions on shallow slopes. If gait cycles exist on very small slopes, those walking motions must also be very slow.

6 Concluding Remarks

The arguments presented in this paper may be applicable, directly or with slight modifications or restrictions, to other passive walkers: these might include walkers with some kinds of springs for energy storage (provided the collision at heelstrike is necessarily dissipative), and even walkers in 3-D which have non-singular mass distributions. For example, for 22-D walkers with (say) springs at the hip, if we fix the spring stiffness and then consider vanishingly small slopes, the arguments of this paper hold without any changes (provided the walking motions follow the basic description).

In a companion paper (Garcia et al. (1998b)), we explore the consequences of the fact proved in this paper.

[5] Otherwise, in the limit as $\epsilon \to 0$, the walker would have a solution where it covered essentially no distance while it shifted from foot to foot, bending alternate knees through relatively large angles. Kneestrike collision losses, which we have not discussed at all in our proof, would preclude such a motion.

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